KVAZICHIZIQLI ISSIQLIK O‘TKAZUVCHANLIK TENGLAMASI UCHUN FARQ MASALASI

ORIFOV ASADBEK DILSHODJON O’G’LI
FARG’ONA DAVLAT UNIVERSITETI AMALIY MATEMATIKA VA AXBOROT TEXNOLOGIYALARI KAFEDRASI 25-AMALIY MATEMATIKA MUTAXASISLIGI MAGISTRANTI
asadbekorifov710235@gmail.com

ANNOTATSIYA: Mazkur maqolada kvazichiziqli issiqlik o‘tkazuvchanlik tenglamasi uchun farq masalasini qo‘yish va uni yechishning samarali sonli usullari ko‘rib chiqilgan. Issiqlik o‘tkazuvchanlik koeffitsienti yechimga bog‘liq bo‘lgan hollarda yuzaga keladigan kvazichiziqli tenglamalar uchun aniq analitik yechimlarni topish qiyin bo‘lgani sababli, farq sxemalaridan foydalanish zarurati tug‘iladi. Maqolada fazo va vaqt bo‘yicha diskretlashtirish asosida farq sxemalari qurilgan, ularning yaqinlashtirish xatosi baholangan hamda barqarorlik shartlari tekshirilgan. Nochiziqli farq tenglamalar sistemasini yechishda Nyuton iteratsion usulining qo‘llanilishi va uning tez yaqinlashuvchanligi ko‘rsatib berilgan. Olingan natijalar issiqlik almashinuvi jarayonlarini sonli modellashtirishda amaliy ahamiyatga ega.

KALIT SO‘ZLAR: Kvazichiziqli issiqlik o‘tkazuvchanlik tenglamasi, farq masalasi, ayirmali sxema, noaniq farq sxemalari, barqarorlik, yaqinlashuvchanlik, yaqinlashtirish xatosi, energetik tengsizliklar usuli, Nyuton iteratsion usuli, sonli modellashtirish.

KIRISH

Matematik fizikaning ko‘plab muhim masalalari issiqlik, massa va energiya almashinuvi jarayonlarini tavsiflovchi differensial tenglamalar bilan ifodalanadi. Ushbu tenglamalar orasida kvazichiziqli issiqlik o‘tkazuvchanlik tenglamalari alohida o‘rin egallaydi, chunki ularda issiqlik o‘tkazuvchanlik koeffitsienti yechimning o‘ziga bog‘liq bo‘lib, bu tenglamalarning nochiziqli xarakterga ega bo‘lishiga olib keladi. Bunday tenglamalar issiqlik tarqalishi, faza o‘tish jarayonlari, energetik qurilmalar va turli texnologik tizimlarni modellashtirishda keng qo‘llaniladi.

Kvazichiziqli issiqlik o‘tkazuvchanlik tenglamalari uchun aniq analitik yechimlarni topish, odatda, faqat ayrim maxsus holatlar bilangina cheklanadi. Umumiy holatda esa ushbu masalalarni yechishda sonli usullardan foydalanish zarur bo‘ladi. Sonli usullar orasida farq (ayirmali) usullar o‘zining soddaligi, universalligi va yuqori aniqlikka erishish imkoniyati bilan ajralib turadi. Biroq kvazichiziqli tenglamalarda nochiziqlik tufayli farq sxemalarining barqarorligi va yaqinlashuvchanligini ta’minlash alohida e’tibor talab qiladi.

So‘nggi yillarda kvazichiziqli issiqlik o‘tkazuvchanlik tenglamalari uchun barqaror va yuqori aniqlikka ega bo‘lgan noaniq va yarim noaniq farq sxemalarini qurish hamda ularni yechishda iteratsion usullardan foydalanish dolzarb masalalardan biri bo‘lib qolmoqda. Xususan, Nyuton iteratsion usuli nochiziqli farq tenglamalar sistemasini samarali va tez yaqinlashuvchi algoritmlar yordamida yechish imkonini beradi.

Mazkur maqolada kvazichiziqli issiqlik o‘tkazuvchanlik tenglamasi uchun farq masalasini qo‘yish, farq sxemalarini qurish, ularning yaqinlashtirish xatosini baholash hamda barqarorlik shartlarini o‘rganish masalalari ko‘rib chiqiladi. Olingan nazariy natijalar issiqlik almashinuvi jarayonlarini sonli modellashtirishda qo‘llanishi mumkin.1. Farq masalasini qo’yish va yaqinlashtirish xatosini hisoblash. Bir jinsli arqon tebranishlarining tenglamasini ko’rib chiqamiz:

$$\frac{\partial^2 u}{\partial t_1^2} = a^2 \frac{\partial^2 u}{\partial x_1^2} + f_1(x_1, t_1), \quad 0 < x_1 < l, \quad t_1 > 0.$$

O’lchamsiz o’zgaruvchilarni $x = x_1 / l$, $t = at_1 / l$ kiritib, ushbu tenglamani quyidagi ko’rinishda qayta yozamiz:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+f(x,t),\quad 0<x<1,\quad 0<t\le T. \qquad (1)$$

Boshlang'ich momentda shartlar berilgan:

$$u(x,0)=u_0(x),\quad \frac{\partial u(x,0)}{\partial t}=\tilde{u}_0(x) \qquad (2)$$

(boshlang'ich chetlanish $u_0(x)$ va boshlang'ich tezlik $\tilde{u}_0(x)$). Arqon uchlariga berilgan qonunlar bo'yicha harakat qiladi:

$$u(0,t)=\mu_1(t),\quad u(1,t)=\mu_2(t). \qquad (3)$$

$\bar{D}=\{0\le x\le 1, 0\le t\le T\}$ sohada to'rtburchaklar tarmoq $\omega_{h\tau}$ ni kiritamiz (§1, 2-bandga o'xshab). (1) tenglama t bo'yicha ikkinchi hosilani o'z ichiga olgani sababli, qatlamlar soni uchtadan kam bo'lmasligi kerak. Yuqoridagi kabi, quyidagi belgilardan foydalanamiz:

$$y=y^j,\quad \hat{y}=y^{j+1},\quad \check{y}=y^{j-1},\quad y_t=\frac{\hat{y}-y}{\tau},\quad y_{\bar{t}}=\frac{y-\check{y}}{\tau},\quad \Delta y=y_{x\bar{x}},$$

$$y_{t\bar{t}}=\frac{y_t-y_{\bar{t}}}{\tau}=\frac{\hat{y}-2y+\check{y}}{\tau^2},\quad y_{\mathring{t}}=\frac{y_t+y_{\bar{t}}}{2}=\frac{\hat{y}-\check{y}}{2\tau}.$$

(1) tenglamaga kiruvchi hosilalarni quyidagi formulalar bilan almashtiramiz:

$$\frac{\partial^2 u}{\partial t^2}\sim u_{t\bar{t}},\quad \frac{\partial^2 u}{\partial x^2}\sim \Delta u=u_{x\bar{x}},\quad f\sim\varphi.$$

Og'irlik koeffitsientli sxemalar oilasini ko'rib chiqamiz:

$$\begin{aligned}&y_{t\bar{t}}=\Delta(\sigma\hat{y}+(1-2\sigma)y+\sigma\check{y})+\varphi,\quad \varphi=f(x,t),\\ &y_0=\mu_1(t),\quad y_N=\mu_2(t),\quad y(x,0)=u_0(x),\quad y_t(x,0)=\tilde{u}_0(x),\end{aligned} \qquad (4)$$

bu yerda $\tilde{u}_0(x)$ keyinroq aniqlanadi.

Chegaraviy shartlar va birinchi boshlang'ich shart $u(x,0)=u_0(x)$ $\omega_{h\tau}$ tarmog'ida aniq qanoatlantiriladi. $\tilde{u}_0(x)$ ni $\tilde{u}(x)-\partial u(x,0)/\partial t=u_t(x,0)-\tilde{u}_0(x)$ yaqinlashtirish xatosi $O(\tau^2)$ miqdorida bo'lishi uchun tanlaymiz. Quyidagi formuladan:

$$\begin{aligned}
u_t(x,0)&=\dot{u}(x,0)+0.5\tau\ddot{u}(x,0)+O(\tau^2)=\\
&=\tilde{u}_0(x)+0.5\tau(u_{xx}(x,0)+f(x,0))+O(\tau^2)=\\
&=\tilde{u}_0(x)+0.5\tau(u_{0''}(x)+f(x,0))+O(\tau^2)
\end{aligned}$$

ko'rinib turibdiki, agar

$$\tilde{u}_0(x)=\tilde{u}_0(x)+0.5\tau(u_{0''}(x)+f(x,0)) \tag{5}$$

deb olsak, $u_t(x,0)-\tilde{u}_0(x)=O(\tau^2)$ bo'ladi.

Shunday qilib, (4) – (5) farq masalasi qo'yilgan bo'ladi. $\hat{y}=y^{j+1}$ ni aniqlash uchun (4) dan chegaraviy masala hosil bo'ladi:

$$\sigma\gamma^2(y_{i+1}^{j+1}+y_{i-1}^{j+1})-(1+2\sigma\gamma^2)y_i^{j+1}=-F_i,\quad 0<i<N,\quad y_0=\mu_1,\quad y_N=\mu_2,$$

bu yerda

$$\gamma=\tau/h,\quad F_i=(2y_i^j-y_i^{j-1})+\tau^2(1-2\sigma)\Delta y^j+\sigma\tau^2\Delta y^{j-1}+\tau^2\varphi,$$

u haydash (progonka) usuli bilan yechiladi. $\sigma>0$ bo'lganda haydash barqaror bo'ladi (I bob, §2, 6-bandga qarang).

Sxema (4) ning yaqinlashtirish xatosini $\varphi=f(x,t)$ bo'lganda hisoblash. Aytaylik, y (4)-(5) masala yechimi, $u=u(x,t)$ (1)-(3) masala yechimi bo'lsin. $y=z+u$ ni(4)ga qo'ysak, quyidagini hosil qilamiz:

$$\begin{aligned}
&z_{t\bar{t}}=\Delta(\sigma\hat{z}+(1-2\sigma)z+\sigma\check{z})+\psi,\\
&z_0=z_N=0,\quad z(x,0)=0,\quad z_t(x,0)=v(x),
\end{aligned} \tag{6}$$

bu yerda

$$\psi=\Delta(\sigma\hat{u}+(1-2\sigma)u+\sigma\check{u})+\varphi-u_{t\bar{t}}$$

— sxema (4) ning $u = u(x,t)$ yechimdagi yaqinlashtirish xatosi, $v = \tilde{u}_0(x) - u_t(x,0)$ — ikkinchi boshlang'ich $y_t = \tilde{u}_0(x)$ shart uchun yaqinlashtirish xatosi. Oldingi ma'lumotlardan $v = O(\tau^2)$ ekani ma'lum.

$\hat{u} = u + \tau u_t$, $\check{u} = u - \tau u_{\bar{t}}$ ekanligini hisobga olsak:

$$\sigma\hat{u} + (1-2\sigma)u + \sigma\check{u} = u + \sigma\tau^2 u_{t\bar{t}},$$

ya'ni

$$\psi = \Delta u + \sigma\tau^2\Delta u_{t\bar{t}} + \varphi - u_{t\bar{t}} = Lu + \sigma\tau^2 L\ddot{u} + f - \ddot{u} + O(\tau^2 + h^2), \tag{7}$$

$\psi = O(\tau^2 + h^2)$ har qanday σ doimiysi uchun (σ τ va h ga bog'liq emas).

Aytaylik, $\sigma = \bar{\sigma} - h^2/(12\tau^2)$, bu yerda $\bar{\sigma}$ — h va τ dan bog'liq bo'lmagan doimiy, (4) sxema barqaror bo'lishi uchun tanlangan ($\bar{\sigma} \geq 1/(4(1-\epsilon))$ talab qilish kifoya, chunki sxema $\sigma \geq 1/(4(1-\epsilon)) - 1/(4\gamma^2), \quad \gamma = \tau/h, \quad \epsilon > 0$ da barqaror).

U holda, agar

$$\varphi = f + \frac{h^2}{12} f_{xx} \tag{8}$$

bo'lsa, (4) sxema yuqori tartibli yaqinlashtirishga ega bo'ladi, $\psi = O(h^4 + \tau^2)$.

Uchinchi turdagi chegaraviy shartlar

$$\frac{\partial u(0,t)}{\partial x} = \beta_1 u(0,t) - \mu_1(t),$$

$$-\frac{\partial u(1,t)}{\partial x} = \beta_2 u(1,t) - \mu_2(t)$$

quyidagi farq tenglamalari bilan yaqinlashtiriladi:

$$\rho_1 y_{t\bar{t}} = \Lambda^-(\sigma\hat{y} + (1-2\sigma)y + \sigma\check{y}) + \varphi^-, \quad i = 0,$$

$$\rho_2 y_{t\bar{t}} = \Lambda^+(\sigma\hat{y} + (1-2\sigma)y + \sigma\check{y}) + \varphi^+, \quad i = N,$$

bu yerda

$$\Lambda^{-} y = \frac{y_x - \beta_1 y}{0.5h}, \quad \Lambda^{+} y = \frac{-\bar{\bar{y}}_x + \beta_2 y}{0.5h},$$

$$\varphi^{-} = \rho_1 \varphi + \frac{\mu_1}{0.5h}, \quad \varphi^{+} = \rho_2 \varphi + \frac{\mu_2}{0.5h}.$$

Bunda, agar

$$\varphi = f(x,t), \quad \rho_1 = \rho_2 = 1, \quad v_1 = \mu_1(t), \quad v_2 = \mu_2(t)$$

bo’lsa, chegaraviy shartlarni yaqinlashtirish xatosi $O(\tau^2 + h^2)$ miqdorda bo’ladi. Agar

$$\sigma = -\frac{h^2}{12\tau^2} + \bar{\sigma}, \quad \bar{\sigma} = \text{const}, \quad \rho_1 = 1 + \frac{h\beta_1}{3}, \quad \rho_2 = 1 + \frac{h\beta_2}{3},$$

$$\varphi = f(x,t) + \frac{h^2}{12} f_{xx},$$

$$v_1(t) = \mu_1(t) + \frac{h^2}{6}\left(\frac{\ddot{\mu}_1(t)}{2} + f_x(0,t) - \beta_1 f(0,t)\right),$$

$$v_2(t) = \mu_2(t) + \frac{h^2}{6}\left(\frac{\ddot{\mu}_2(t)}{2} - f_x(1,t) - \beta_2 f(1,t)\right),$$

bo’lsa, $O(h^4 + \tau^2)$ aniqlikdagi sxemani hosil qilamiz. Bu sxema boshlang’ich tenglamani $x = 0$, $x = 1$ tugunlarida $O\left(\frac{\tau^2 + h^4}{h}\right)$ xato bilan, chegaraviy shartlarni esa $O(\tau^2 + h^4)$ xato bilan yaqinlashtiradi.

2. Barqarorlikni tekshirish. Keling, sxema (4) ning boshlang’ich ma’lumotlarga nisbatan barqarorligini o’rganamiz (bir jinsli chegaraviy shartlar va tenglamaning nol o’ng qismi uchun). Buning uchun quyidagi masalani ko’rib chiqamiz:

$$\begin{aligned} &y_{t\bar{t}} = \Delta(\sigma \hat{y} + (1 - 2\sigma) y + \sigma \check{y}) = \Delta y^{(\sigma)}, \\ &y_0 = y_N = 0, \quad y(x,0) = u_0(x), \quad y_t(x,0) = \tilde{u}_0(x). \end{aligned} \tag{4a}$$

Uning yechimini o'zgaruvchilarni ajratish usuli bilan topamiz. Buning uchun, §1, 4-bandga o'xshab, $y(x,t) = X(x)T(t) \neq 0$ ko'rinishidagi maxsus yechimlarni qidiramiz. $y = XT$ ni (4a) tenglamaga qo'yganimizdan so'ng:

$$\frac{\Delta X}{X} = \frac{T_{t\bar{t}}}{T^{(\sigma)}} = -\lambda. \tag{9}$$

Bu yerdan va $y_0 = y_N = 0$ chegaraviy shartlaridan $X(x)$ uchun o'ziga xos qiymatlar masalasini hosil qilamiz:

$$\Delta X + \lambda X = 0, \quad x \in \omega_h, \quad X(0) = X(1) = 0, \quad X(x) \neq 0.$$

Uning yechimlari mavjud:

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\pi kh}{2}, \quad X^{(k)}(x) = \sqrt{2}\sin \pi kx.$$

(9) dan $T_k(t)$ uchun ikkinchi tartibli farq tenglamasini hosil qilamiz:

$$(T_k)_{t\bar{t}} + \lambda_k T_k^{(\sigma)} = 0,$$

yoki

$$(1+\sigma\tau^2\lambda_k)\hat{T}_k - 2(1+(\sigma-0.5)\tau^2\lambda_k)T_k + (1+\sigma\tau^2\lambda_k)\check{T}_k = 0,$$

uni quyidagi ko'rinishda qayta yozamiz:

$$\hat{T}_k - 2(1-\alpha_k)T_k + \check{T}_k = 0, \quad \alpha_k = \frac{0.5\tau^2\lambda_k}{1+\sigma\tau^2\lambda_k}. \tag{10}$$

Bu tenglamaning yechimini $T_k = T_k(t_j) = q_k^j$ ko'rinishida qidiramiz. (10) dan q uchun kvadrat tenglamani topamiz: $q^2 - 2(1-\alpha)q + 1 = 0$ (k indeksini vaqtincha tushirib qo'yamiz). Uning ildizlari $q_{1,2} = 1 - \alpha \pm \sqrt{\alpha^2 - 2\alpha}$ ga teng. Agar $0 < \alpha < 2$ bo'lsa, $q_{1,2} = 1 - \alpha \pm i\sqrt{\alpha(2-\alpha)}$ ildizlar kompleks bo'lib, $|q_{1,2}| = 1$ bo'ladi. Yangi φ_k o'zgaruvchini,

$$\cos\varphi_k = 1 - \alpha_k, \quad \sin\varphi_k = \sqrt{\alpha_k(2-\alpha_k)}.$$

deb olib kiritamiz. U holda $q_1^{(k)} = e^{i\varphi_k}$, $q_2^{(k)} = e^{-i\varphi_k}$ bo'ladi. (10) tenglamaning umumiy yechimi quyidagi ko'rinishga ega:

$$T_k(t_j) = C_k(q_1^{(k)})^j + D_k(q_2^{(k)})^j = A_k \cos j\varphi_k + B_k \sin j\varphi_k,$$

bu yerda A_k va B_k — ixtiyoriy doimiylar.

(4a) masala yechimini maxsus yechimlar yig'indisi ko'rinishida qidiramiz:

$$y^j = \sum_{k=1}^{N-1}(A_k \cos j\varphi_k + B_k \sin j\varphi_k)X^{(k)}(x). \tag{11}$$

$u_{0,k}$ va $\tilde{u}_{0,k}$ larni $u_0(x)$ va $\tilde{u}_0(x)$ funksiyalarning yoyilma koeffitsientlari deb faraz qilaylik:

$$u_0(x) = \sum_{k=1}^{N-1} u_{0,k} X^{(k)}(x), \quad \tilde{u}_0(x) = \sum_{k=1}^{N-1} \tilde{u}_{0,k} X^{(k)}(x). \tag{12}$$

Biz (11) ifodaning boshlang'ich shartlar $y^0 = u_0, y_t^0 = (y^1 - y^0)/\tau = \dot{u}_0(x)$ larni qanoatlantirishini talab qilamiz. Shunda A_k va B_k larni aniqlash uchun quyidagi shartlarni olamiz:

$$A_k = u_{0k}, \quad A_k \frac{\cos q_k - 1}{\tau} + B_k \frac{\sin q_k}{\tau} = \dot{u}_{0k}.$$

Bu yerdan topamiz:

$$A_k = u_{0k}, \quad B_k = \frac{1-\cos q_k}{\sin q_k} u_{0k} + \frac{\tau}{\sin q_k} \dot{u}_{0k}. \tag{13}$$

A_k va B_k larni (11) ga qo'yib, aniq o'zgartirishlardan so'ng, bizda:

$$y^j = \sum_{k=1}^{N-1}\left(\frac{\cos(j-0.5)q_k}{\cos 0.5 q_k} u_{0k} + \frac{\tau \sin jq_k}{\sin q_k} \dot{u}_{0k}\right) X^{(k)}(x). \tag{14}$$

Avvalambor, $\sigma = 0$ bo'lgandagi (4a) sxemasi, ya'ni quyidagi sxema uchun $\| y^j \|$ normaning bahosini olamiz:

$$y_{tt} = \Delta y, \quad y_0 = y_N = 0, \quad y(x,0) = u_0(x), \quad y_t(x,0) = \dot{u}_0(x). \tag{15}$$

$\sigma = 0$ da bizda:

$$\alpha_k = 0, \quad 0.5\tau^2\lambda_k = \mu_k, \quad \cos q_k = 1 - \mu_k, \quad \sin q_k = \sqrt{\mu_k(2-\mu_k)}.$$

$\omega_{h\tau}$ to'r qadamlari quyidagi munosabatni qanoatlantirsin deymiz:

$$\frac{\tau^2}{h^2} \le \frac{1}{1+\epsilon}, \tag{16}$$

bu yerda $\epsilon > 0$ — ixtiyoriy son. Shunda:

$$\mu_k \le \frac{2}{1+\epsilon}, \quad k = 1, 2, \ldots, N-1, \tag{17}$$

va demak,

$$\cos 0.5q_k = \sqrt{\frac{1-\mu_k}{2}} \ge \sqrt{\frac{\epsilon}{1+\epsilon}}. \tag{18}$$

Bundan tashqari,

$$\frac{\sin q_k}{\tau} = \frac{2\sin 0.5q_k}{\tau}\cos 0.5q_k \ge \frac{2\sin 0.5q_k}{\tau}\sqrt{\frac{\epsilon}{1+\epsilon}},$$

va chunki

$$\frac{2\sin 0.5q_k}{\tau} = \frac{\sqrt{2(1-\cos q_k)}}{\tau} = \frac{\sqrt{2\mu_k}}{\tau} = \sqrt{\lambda_k},$$

demak, quyidagi baho o'rinli:

$$\frac{\sin q_k}{\tau} \ge \sqrt{\lambda_k \frac{\epsilon}{1+\epsilon}}. \tag{19}$$

(14) dan quyidagi tengsizlik kelib chiqadi:

$$\| y^j \| \le \left\| \sum_{k=1}^{N-1} \frac{\cos(j-0.5)q_k}{\cos 0.5q_k} u_{0k} X^{(k)} \right\| + \left\| \sum_{k=1}^{N-1} \frac{\tau \sin jq_k}{\sin q_k} \dot{u}_{0k} X^{(k)} \right\|,$$

unga (18) va (19) baholarni qo'ysak, bizda:

$$\| y^j \| \le \sqrt{\frac{1+\epsilon}{\epsilon}} \left(\| u_0 \| + \left(\sum_{k=1}^{N-1} \frac{(\dot{u}_{0k})^2}{\lambda_k} \right)^{1/2} \right).$$

Endi shuni ta'kidlaymizki,

$$\left(\sum_{k=1}^{N-1}\frac{(\dot{u}_{0k})^2}{\lambda_k}\right)^{1/2}$$

ifoda H_A^{-1} dagi "salbiy" norma, ya'ni

$$\|\dot{u}_0\|_{A^{-1}}=(A^{-1}\dot{u}_0,\dot{u}_0)^{1/2},$$

ekandan boshqa narsa emas. Bu yerda $Ay=-\Delta y=-y_{\bar{x}x}$ operatori $\bar{\omega}_h$ to'rida berilgan va $x=0,\ x=l$ da nolga teng bo'lgan funksiyalar fazosida ta'riflangan. Darhaqiqat,

$$A^{-1}\dot{u}_0=\sum_{k=1}^{N-1}\dot{u}_{0k}A^{-1}X^{(k)}=\sum_{k=1}^{N-1}\frac{\dot{u}_{0k}}{\lambda_k}X^{(k)},$$

va shuning uchun

$$(A^{-1}\dot{u}_0,\dot{u}_0)=\sum_{k=1}^{N-1}\frac{(\dot{u}_{0k})^2}{\lambda_k}.$$

Shunday qilib, agar (16) shart bajarilsa, (15) sxemasi uchun quyidagi baho o'rinli:

$$\|y^j\|\le\sqrt{\frac{1+\epsilon}{\epsilon}}\left(\|u_0\|+\|\dot{u}_0\|_{A^{-1}}\right). \tag{20}$$

Xuddi shu baho (4a) sxemasi uchun ham o'rinli bo'ladi, agar σ parametri quyidagi shartni qanoatlantirsa:

$$\sigma\ge\frac{1+\epsilon}{4}-\frac{h^2}{4\tau^2}, \tag{21}$$

bu yerda $\epsilon>0$ — ixtiyoriy son. Bunga ishonch hosil qilish uchun yuqoridagi isbotda hamma joyda μ_k ni

$$\alpha_k=\frac{0.5\tau^2\lambda_k}{1+\sigma\tau^2\lambda_k}$$

ga almashtirish kifoya. (4) sxemaning o'ng tomonga nisbatan barqarorligini o'rganish uchun superpozitsiya (ustma-ust qo'yish) printsipini qo'llaymiz. Quyidagi masalani ko'rib chiqaylik:

$$y_{tt} = \Delta y^{(\sigma)} + \varphi,$$
$$y_0 = y_N = 0, \quad y(x,0) = 0, \quad y_t(x,0) = 0. \tag{4b}$$

Uning yechimini quyidagi ko'rinishda qidiramiz:

$$y^j = \sum_{i'=0}^{j-1} \tau Y^{j,i'}, \tag{22}$$

bu yerda $Y^{j,i'}$, $ i' $ belgilangan bo'lganda $ j $ ning funksiyasi sifatida, bir jinsli tenglamani qanoatlantiradi:

$$Y_{tt}^{j,i'} = \Delta\left(\sigma Y^{j+1,i'} + (1-2\sigma)Y^{j,i'} + \sigma Y^{j-1,i'}\right), \quad 0 \le i' < j, \tag{23}$$

chegaraviy shartlarni:

$$Y_0^{j,i'} = Y_N^{j,i'} = 0, \tag{24}$$

va boshlang'ich shartlarni:

$$Y^{i',i'} = 0, \quad Y_t^{i',i'} = \frac{Y^{i'+1,i'} - Y^{i',i'}}{\tau} = \frac{Y^{i'+1,i'}}{\tau} = \Phi^{i'}, \tag{25}$$

qanoatlantiradi. Bu yerda $\Phi^{i'}$ (46) nobir jinsli tenglama bajariladigan qilib tanlanadi.

Φ^j funksiya qanoatlantiradigan tenglamani topamiz. $Y^{j,i'}$ ning ta'rifidan kelib chiqadiki:

$$y_{tt}^j = \frac{1}{\tau}Y^{j+1,j} + \sum_{i'=0}^{j-1} \tau Y_{tt}^{j,i'}, \quad \Delta(y^j)^{(\sigma)} = \sigma\tau\Delta Y^{j+1,j} + \sum_{i'=0}^{j-1} \tau\Delta(Y^{j,i'})^{(\sigma)}.$$

Bu ifodalarni (23) ga qo'yib, topamiz:

$$\frac{1}{\tau}Y^{j+1,j} - \sigma\tau\Delta Y^{j+1,j} = \tau\varphi^j, \tag{26}$$

bundan $\Phi = \Phi^j = Y^{j+1,j} / \tau$ uchun tenglamani olamiz:

$$\Phi - \sigma\tau^2\Delta\Phi = \varphi, \quad \Phi_0 = \Phi_N = 0. \tag{27}$$

Endi (46) masala yechimi y^j ning o'ng tomon φ orqali bahosini olishga o'tamiz. Faraz qilaylik, barqarorlik sharti (21) bajarilsin. U holda (23) masala yechimi uchun (20) baho o'rinli bo'lib, bu holda u quyidagi ko'rinishga ega:

$$\| Y^{j,i'} \| \le \sqrt{\frac{1+\varepsilon}{\varepsilon}} \| Y_t^{i',i'} \|_{A^{-1}} = \frac{1}{\tau}\sqrt{\frac{1+\varepsilon}{\varepsilon}} \| Y^{i'+1,i'} \|_{A^{-1}} .$$

Shuning uchun (22) dan va uchburchaksizlik tengsizligidan foydalanib, quyidagini olamiz:

$$\| y^j \| \le \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{i'=0}^{j-1} \tau \| Y^{i'+1,i'} \|_{A^{-1}} .$$

$\| Y^{i'+1,i'} \|_{A^{-1}}$ ning bahosini (27) tenglamadan olamiz. Φ va $\varphi ni \{X^{(k)}\}$ o'z funksiyalari bo'yicha yoyamiz:

$$\Phi = \sum_{k=1}^{N-1} \Phi_k X^{(k)}, \quad \varphi = \sum_{k=1}^{N-1} \varphi_k X^{(k)}. \tag{28}$$

(28) ni (27) ga qo'ysak, $\Phi_k = \varphi_k / (1+\sigma\tau^2\lambda_k)$ topamiz, shunday qilib $\sigma \ge 0$ da:

$$\| \Phi \|_{A^{-1}}^2 = \sum_{k=1}^{N-1} \frac{\Phi_k^2}{\lambda_k} = \sum_{k=1}^{N-1} \frac{\varphi_k^2}{(1+\sigma\tau^2\lambda_k)^2 \lambda_k} \le \sum_{k=1}^{N-1} \frac{\varphi_k^2}{\lambda_k} = \| \varphi \|_{A^{-1}}^2 ,$$

ya'ni

$$\| Y^{i'+1,i'} \|_{A^{-1}} \le \tau \| \varphi^{i'} \|_{A^{-1}} .$$

Shunday qilib, agar $\sigma \ge 0$ va (21) shart bajarilsa, (46) sxemasi uchun quyidagi baho o'rinli:

$$\| y^j \| \le \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{i'=0}^{j-1} \tau \| \varphi^{i'} \|_{A^{-1}} .$$

Bir jinsli chegaraviy shartlar $y_0 = y_N = 0$ bo'lgan (4) masalasi uchun quyidagi baho o'rinli:

$$\| y^j \| \le \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left(\| y^0 \| + \| y_t^0 \|_{A^{-1}} + \sum_{i'=0}^{j-1} \tau \| \varphi^{i'} \|_{A^{-1}} \right),$$

agar $\sigma \ge 0$ va (21) shart bajarilsa.

Qiziq jihati shundaki, $y^1 = y(\tau)$ ni maxsus tanlash orqali (15) sxemasining $\tau \le h$ sharti ostida L_2 fazoda barqarorligini isbotlash mumkin.

Quyidagi farqlar sxemasini ko'rib chiqaylik:

$$y_{tt}^j = y_{\bar{x}x}^j, \quad j = 1,2,\ldots, \tag{29}$$

$$y^0 = u_0, \quad y_t^0 = \tilde{u}_0 + 0.5\tau y_{\bar{x}x}^0. \tag{30}$$

1-bandga ko'ra, (29) — (30) masala (1) — (2) tenglamani $O(\tau^2 + h^2)$ aniqlikda approksimatsiya qiladi.

y^j yechimni $y^0 = u_0$ va $\tilde{u}_0$ orqali ifodalaymiz. Avvalgidek ishlab, topamiz:

$$y^j = \sum_{k=1}^{N-1}\left(u_{0k}\cos j\varphi_k + \frac{\tau\tilde{u}_{0k}}{\sin\varphi_k}\sin j\varphi_k\right)X^{(k)}, \tag{31}$$

bu yerda $\tilde{u}_{0k}$ — $\tilde{u}_0(x)$ funksiyaning Furye koeffitsientlari, u_{0k}, φ_k miqdorlari esa avvalgi ma'noni saqlaydi.

(31)-ni kvadratga ko'tarib va

$$2\left(\frac{\pi_{bk}}{\sin\varphi_k}\cos j\varphi_k\right)(\mu_{0k}\sin j\varphi_k) \le \tau^2\frac{\pi_{bk}^2}{\sin^2\varphi_k}\cos^2 j\varphi_k + \mu_{0k}^2\sin^2 j\varphi_k$$

baholashdan foydalanib, biz

$$\| y^j \|^2 \le \| u_0 \|^2 + \tau^2\sum_{k=1}^{N-1}\frac{\bar{u}_{0k}^{2}}{\sin^2\varphi_k}$$

ga egamiz. Ifodani quyidan baholaymiz:

$$\frac{\tau^2}{\sin^2\varphi_k} = \frac{1}{\lambda_k(1-\tau^2\lambda_k/4)}. \quad \gamma = \tau/h \le 1$$ bo'lsin, u holda

$$\lambda_k(1-\tau^2\lambda_k/4) =$$

$$\frac{4}{h^2}\sin^2\frac{\pi kh}{2}\left(1-\gamma^2\sin^2\frac{\pi kh}{2}\right) \ge \frac{4}{h^2}\sin^2\frac{\pi kh}{2}\left(1-\sin^2\frac{\pi kh}{2}\right) =$$

$$\frac{4}{h^2}\sin^2\frac{\pi kh}{2}\cos^2\frac{\pi kh}{2} \ge \frac{4}{h^2}\sin^2\frac{\pi kh}{2}\cos^2\frac{\pi kh}{2} = \frac{\sin^2\pi h}{h^2}.$$

II bobda shuni ko'rsatgan edik: agar $h_1 \le 0,5$ bo'lsa, $\frac{4}{h^2}\sin^2\frac{\pi h}{2} \ge 8$. Shuning uchun, $h_1 = 2h$ belgilasak, bizda

$$\frac{\sin^2\pi h}{h^2} = \frac{4}{h_1^2}\sin^2\frac{\pi h_1}{2} \ge 8, \quad h \le \frac{1}{4}$$

bo'ladi.

Shunday qilib, agar $\tau \le h$ va $h \le \frac{1}{4}$ bo'lsa, (29) – (30) masala yechimi uchun biz quyidagi bahoga egamiz:

$$\| y' \|^2 \le \| u_0 \|^2 + \frac{1}{8}\| u_0 \|^2.$$

FOYDALANILGAN ADABIYOTLAR RO'YXATI:

1. Samarskiy A.A., Gulin A.V. Chislennye metody (Sonli usullar). – M.: Nauka, 1989. – 432 s.

2. Samarskiy A.A., Nikolaev E.S. Metody resheniya setochnyx uravneniy (Setka tenglamalarini yechish usullari). – M.: Nauka, 1978. – 592 s.

3. Richtmyer R.D., Morton K.W. Difference Methods for Initial-Value Problems (Boshlang'ich qiymat masalalari uchun ayirma usullari). – NY: Interscience, 1967.

4. Godunov S.K., Ryabenkiy V.S. Vvedenie v teoriyu raznostnyx skhem (Ayirma sxemalar nazariyasiga kirish). – M.: Fizmatgiz, 1962.

5. Marchuk G.I. Metody vychislitel'noy matematiki (Hisoblash matematikasi usullari). – M.: Nauka, 1980.

6. A.A.Samarskiy. Teoriya raznostnyx skhem (Ayirma sxemalar nazariyasi). – M.: Nauka, 1977.