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MODELING GEODESIC LINES IN CURVED SPACETIME USING DIFFERENTIAL GEOMETRY WITHIN THE FRAMEWORK OF GENERAL RELATIVITY

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Abstract: This paper investigates the modeling of geodesic lines in curved spacetime using methods of differential geometry as applied within the framework of General Relativity. By employing Riemannian geometry, we describe how free-falling particles move along geodesics under gravitational influence, and how spacetime curvature governs such motion. The study focuses on the mathematical structure of Lorentzian manifolds, the computation of Christoffel symbols, and the formulation of geodesic equations in Schwarzschild and Friedmann–Lemaître–Robertson–Walker (FLRW) metrics. Symbolic computations were performed to derive geodesic trajectories and analyze curvature effects. The outcomes demonstrate the effectiveness of differential geometry in visualizing gravitational dynamics and deepen the mathematical understanding of Einstein's field equations. These results can support both theoretical research and computational physics education.

Keywords: Differential geometry, general relativity, geodesics, curved spacetime, riemannian manifold, christoffel symbols, schwarzschild metric, lorentzian geometry

1. INTRODUCTION

General Relativity (GR), introduced by Albert Einstein in 1915, revolutionized the understanding of gravitation by interpreting it as a manifestation of spacetime curvature rather than a traditional force [1]. Within GR, differential geometry plays a fundamental role in defining the geometric structure of spacetime, particularly through the use of Riemannian and Lorentzian manifolds [2], [3]. A core concept in this framework is the geodesic line, which represents the trajectory of a free-falling particle in the absence of non-gravitational forces [4]. The path of such a particle is determined not by Newtonian dynamics but by the curvature of the underlying spacetime itself. Hence, to compute or model these paths, one must employ differential geometric tools such as Christoffel symbols, affine connections, and curvature tensors [5]. This research aims to model geodesic lines in selected curved spacetime metrics using symbolic computation and tensor calculus, illustrating how these trajectories can be derived and visualized in the context of Einstein's theory. This investigation not only bridges abstract mathematical theory and physical interpretation but also aids in educational visualization and simulation tools for modern physics curricula [6], [7].

2. LITERATURE REVIEW

Previous works have thoroughly established the mathematical foundations of General Relativity using Riemannian geometry. For example, Wald [8] and Misner, Thorne, and Wheeler [9] provide comprehensive treatments of Lorentzian manifolds and geodesic motion. Their contributions laid the groundwork for modern computational relativity. Chandrasekhar's work on Schwarzschild geometry [10] provided deep insights into the analytic properties of spacetime around spherically symmetric masses. More recent studies by Baez and Muniain [11] show how differential geometry can be applied computationally for simulating geodesic motion in dynamic spacetimes. In terms of educational applications, GeoGebra and symbolic software like Mathematica have been increasingly used to visualize geodesic paths in Schwarzschild or FLRW backgrounds [12]. Meanwhile, research by Padmanabhan [13] emphasizes the pedagogical need for integrating differential geometry with physical interpretations of Einstein's equations. These works guide the present research, which combines analytical derivation with computational modeling of geodesic equations, allowing for both academic and practical impact.

3. METHODOLOGY

The methodology of this study consists of the following key steps:

1. **Mathematical Framework:** We begin by defining the spacetime as a 4-dimensional Lorentzian manifold (M, g), where g is the metric tensor. The Christoffel symbols Γ^i_{jk} are derived from the metric using:

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}\left(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}\right)$$

2. **Geodesic Equation:** Geodesic lines are computed from the second-order differential equation:

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0$$

3. **Metric Selection:** We perform case studies using:

- Schwarzschild metric (for black hole geometry)
- FLRW metric (for cosmological expansion)

4. **Computation:** Using symbolic tools (e.g., Mathematica), Christoffel symbols and geodesic equations are computed, and trajectories are simulated for both metrics.

Table 1. **Selected Metrics and Their Key Properties**

Metric Type	**Geometry Description**	**Coordinates Used**	**Symmetry**
Schwarzschild	Static, spherically symmetric mass	(t, r, θ, ϕ)	Time-independent
FLRW	Homogeneous, isotropic universe	(t, r, θ, ϕ)	Dynamic expansion

4. RESULTS AND RECOMMENDATIONS

Results:

- Geodesic simulations show how test particles spiral or fall into gravitational wells in Schwarzschild spacetime.
- FLRW metric analysis shows radial geodesics expanding over time, reflecting cosmic scale factors.
- Visualizations clarify how curvature leads to acceleration-like effects in purely geometric terms.

Recommendations:

- Differential geometry should be more deeply integrated into physics education for relativity.
- Open-source tools should be developed to simulate geodesics interactively.

- Similar modeling approaches can be extended to rotating (Kerr) or charged (Reissner–Nordström) spacetimes.

CONCLUSION

This study demonstrates the practical use of differential geometry in modeling geodesic motion under General Relativity. By applying tensor calculus to Schwarzschild and FLRW metrics, we have shown how curvature dictates particle motion without the need for external forces. The results validate the predictive power of Einstein's theory and support its mathematical formalism. Furthermore, the use of computational tools to derive and visualize these paths underscores the importance of interdisciplinary integration between geometry, physics, and computation. These insights are crucial not only for theoretical development but also for improving educational and research tools in modern gravitational physics.

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